MATH 141: Midterm 1

Name: Key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
		50

Conceptual understanding

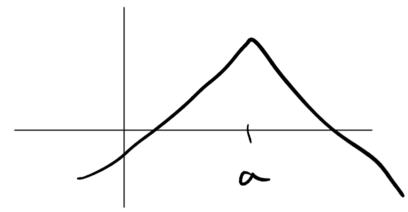
execution

common pitfalls

1. Short answer questions:

   (a) If a function is continuous at $x = a$, is it differentiable at $x = a$? Explain for full credit.

No. A function with a corner/cusp at $x = a$ is continuous but not differentiable.



   (b) Suppose

$$\lim_{x \to 2^+} f(x) = 2.00001 \qquad \lim_{x \to 2^-} f(x) = 2$$

   Is it true that $\lim_{x \to 2} f(x) = 2$?

No. $\lim_{x \to 2} f(x)$ exists if and only if

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

   (c) Suppose you evaluate a limit

$$\lim_{x \to -5} \frac{f(x)}{g(x)} = \cdots = \frac{0}{0}$$

   What global factor do you need to generate in the numerator/denominator to cancel?

$(x+5)$. Plug in $x = -5$. It's 0.

Note: You might wonder for a limit like

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\sin(0)}{0} = \frac{0}{0}$$

how is it possible to create a global factor of x in the numerator. Calculus II will show you how (called Taylor Series).

2. Perform the given instruction. Remember to use the relevant laws/properties and **fully simplify**.

   Not simplifying = lose points.

   (a) Completely simplify: $\dfrac{\frac{1}{x} - \frac{1}{x+h}}{h}$

   Compound fraction. Focus on numerator as a subproblem

$$\frac{\frac{x+h}{x+h}\cdot\frac{1}{x} - \frac{1}{(x+h)}\cdot\frac{x}{x}}{h} = \frac{\frac{x+h}{(x+h)x} - \frac{x}{(x+h)x}}{h}$$

$$= \frac{\frac{x+h-x}{(x+h)x}}{h}$$

$$= \frac{\frac{h}{(x+h)x}}{h} \leftarrow \frac{\frac{a}{b}}{c} = \frac{a}{b}\cdot\frac{1}{c}$$

Fraction Law 2.

$$= \frac{\cancel{h}}{(x+h)x}\cdot\frac{1}{\cancel{h}}$$

Fraction Law 5.

$$= \boxed{\frac{1}{(x+h)x}}$$

← keep factored. global factors are more simplified than global terms.

   (b) Expand: $(3x^2+1)(4x^4-2x) - (16x^3-2)(x^3+x)$

Critical error!!! Subtracting four terms.

Need to dist − to all terms. Also called parentheses.

$$(3x^2+1)(4x^4-2x) - (16x^3-2)(x^3+x)$$

$$= 12x^6 - 6x^3 + 4x^4 - 2x - \left(16x^6 + 16x^4 - 2x^3 - 2x\right)$$

$$= 12x^6 - 6x^3 + 4x^4 - 2x - 16x^6 - 16x^4 + 2x^3 + 2x$$

$$= -4x^6 - 12x^4 - 4x^3$$

$$= \boxed{-4x^3(x^3+3x+1)}$$

(c) Rationalize the numerator: $\dfrac{\sqrt{x-h}-\sqrt{x}}{h}$

$$\overset{A}{}\quad - \quad \overset{B}{} \qquad \overset{A}{} \quad + \quad \overset{B}{} \qquad A^2 - B^2$$

$$\frac{\sqrt{x-h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x-h}+\sqrt{x}}{\sqrt{x-h}+\sqrt{x}} = \frac{\left(\sqrt{x-h}\right)^2 - \left(\sqrt{x}\right)^2}{h \cdot \left(\sqrt{x-h}+\sqrt{x}\right)}$$

h multiplies into 2 global terms

$$= \frac{x-h-x}{h \cdot \left(\sqrt{x-h}+\sqrt{x}\right)}$$

$$= \frac{-\cancel{h}}{\cancel{h}\left(\sqrt{x-h}+\sqrt{x}\right)}$$

$$= \boxed{\frac{-1}{\sqrt{x-h}+\sqrt{x}}}$$

(d) Simplify: $\dfrac{(2x^3-x)^5(x-4)^3-(2x^3-x)^4(x-4)^2}{(2x^3-x)^6}$

When I see this word, I always first ask "can I factor?" because factors are more simplified than terms.

Therefore, the numerator is a two term factorization problem. Lecture Note III says when faced with 2 terms, try GCF first!

$$\frac{(2x^3-x)^5(x-4)^3-(2x^3-x)^4(x-4)^2}{(2x^3-x)^6} = \frac{\cancel{(2x^3-x)^4}(x-4)^2\left((2x^3-x)(x-4)-1\right)}{\cancel{(2x^3-x)^6}\,(2x^3-x)^2}$$

$$= \boxed{\frac{(x-4)^2\left(2x^4-8x^3-x^2+4x-1\right)}{(2x^3-x)^2}}$$

this means you must pass VLT.

3. Draw the $\boxed{\text{graph of a function}}$ which satisfies the following:

(a) $f(-2) = 2$

(b) $f(2) = -2$

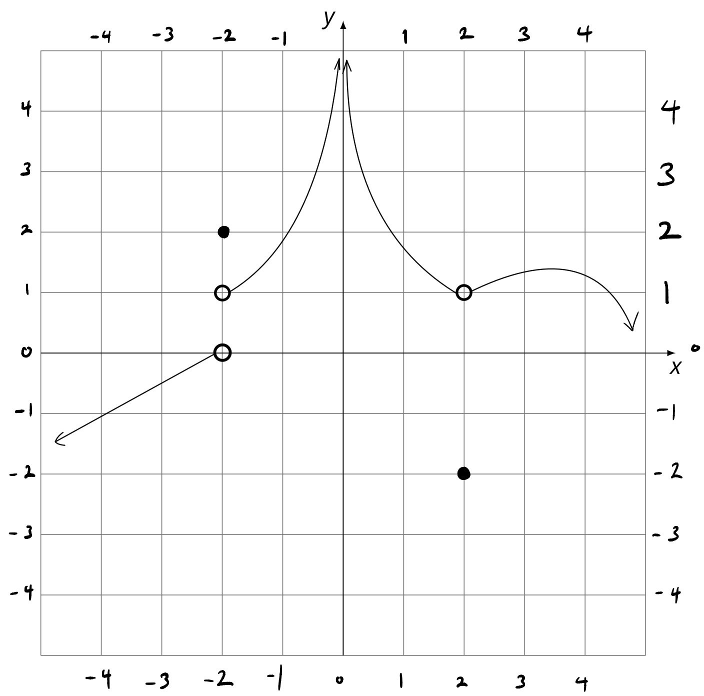
(c) $\lim_{x \to 2} f(x) = 1$

(d) $\lim_{x \to -2^-} f(x) = 0$

(e) $\lim_{x \to -2^+} f(x) = 1$

(f) $\lim_{x \to 0} f(x) = \infty$

Answers may vary.



4. Suppose $f(x) = 3x^2 - x$.

(a) What is the limit definition of the derivative $f'(x)$? Write it down.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Find $f'(x)$ for the given function $f(x)$. You must use the limit definition to receive credit.

$(x+h)$ is plugged into every x in

$f(x) = 3x^2 - x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - (x+h) - (3x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - x - h - 3x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - x - h - 3x^2 + x}{h}$$

every term w/o h as a factor cancels

$$= \lim_{h \to 0} \frac{6xh + h^2 - h}{h}$$

allowing you to create a global factor of h, to cancel!

the only calculus step. continuity

$$\overset{\smile}{=} \lim_{h \to 0} \frac{\cancel{h}(6x + h - 1)}{\cancel{h}} = \lim_{h \to 0} 6x + h - 1 = 6x + 0 - 1$$

$$= \boxed{6x - 1}$$

(c) Find the equation of the tangent line of $f(x)$ at the point $(1, 2)$.

Equation is

$$y - 2 = f'(1) \cdot (x - 1)$$

$f'(x) = 6x - 1$ from (b)

$$y - 2 = (6 \cdot 1 - 1)(x - 1)$$

careful. mult 2 terms.

$$y - 2 = 5(x - 1)$$

$$y = 5x - 5 + 2 \to \boxed{y = 5x - 3}$$

5. Find the derivative of the following functions. You may use formulas.

(a) $f(x) = 999$

$$f'(x) = 0$$

(b) $g(a) = -a^2$

$$g'(a) = \frac{d}{da}\left[-a^2\right] = -\frac{d}{da}\left[a^2\right]$$

$$= -2a^{2-1}$$

$$= \boxed{-2a}$$

(c) $f(x) = 4x^3 - 2x^2 + x - 5$

$$f'(x) = 4 \cdot 3x^{3-1} - 2 \cdot 2x^{2-1} + 1 \cdot x^{1-1} - 0$$

$$= \boxed{12x^2 - 4x + 1}$$

(d) $g(x) = \dfrac{x^2 \cdot \sqrt{x} \cdot x^{3/2} + 1}{x}$ ← num is 2 term ← denom is 1 factor.

You should simplify because you can.

$$= \frac{x^2 \cdot x^{\frac{1}{2}} \cdot x^{\frac{3}{2}} + 1}{x}$$

reverse Fraction Law 3. Lecture Note I

$$= \frac{x^4}{x} + \frac{1}{x}$$

$$= x^3 + x^{-1}$$

$$g'(x) = \frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[x^{-1}\right] = 3x^2 - x^{-2} = \boxed{3x^2 - \frac{1}{x^2}}$$